1 [7 points]. Solve the initial value problem

$$z'' - 3z' + 2z = 2e^{3x}, \qquad z(0) = 1, \qquad z'(0) = 0.$$

SOLUTION:
Homogeneous solution: The characteristic equation is

$$Q(r) = r^2 - 3r + 2 = (r-1)(r-2).$$

Thus, $z_h = C_1 e^x + C_2 e^{2x}$.

Particular Solution: Let $\mathcal{L} = z'' - 3z' + 2z$. Note $Q(3) = 9 - 9 + 2 = 2$. Hence, using the "fast" method, take $r = 3$ and $m = 0$. We get

$$2e^{3x} = \mathcal{L}\left(A\frac{x^m}{m!}e^{3x}\right) = Ae^{3x}Q(3).$$

So take $A = 1$. This gives $z_p = e^{3x}$.

Initial Conditions: We have from above $z = z_h + z_p = C_1 e^x + C_2 e^{2x} + e^{3x}$. Thus,

$$z(0) = 1 \Rightarrow \quad 1 = C_1 + C_2 + 1$$

and

$$z'(0) = 0 \Rightarrow \quad 0 = C_1 + 2C_2 + 3.$$

So we find $C_2 = -3$ and $C_1 = 3$.

So the solution to the IVP is $\boxed{z = 3e^x - 3e^{2x} + e^{3x}}$

2a [6 points]. Find solution $y_2(t)$ of

$$(t^2 - 1)y'' - 2ty' + 2y = 0$$

where one of the solutions is $y_1(t) = t$ and solution y_2 is such that $W(y_1, y_2) = -1$ at $t = 0$ and $y_2(0) = 1$.

SOLUTION:

Step 1: Abel's theorem gives $W = \exp\left(\int \frac{2t}{t^2-1}\right) = C(t^2 - 1)$. Since we want $W(y_1, y_2)(0) = -1$, we must have $C = 1$.

Step 2: Find y_2.

By definition of the Wronskian, $W = \det\begin{pmatrix} t & y_2 \\ 1 & y_2' \end{pmatrix} = ty_2' - y_2$. So we obtain the linear ODE

$$ty_2' - y_2 = t^2 - 1 \quad \Rightarrow \quad y_2' - \frac{1}{t}y_2 = \frac{t^2 - 1}{t}.$$

An integrating factor is $\mu(t) = \exp(\int \frac{-1}{t})$. So

$$\frac{1}{t}y_2 = \int \frac{t^2-1}{t^2} = t + \frac{1}{t} + C.$$

Hence, $y_2 = t^2 + 1 + Ct$.

Step 3: Initial Conditions. We want $y_2(0) = 1$, so we may take $C = 0$.

So a solution is $\boxed{y_2 = t^2 + 1}$

2b [6 points]. Find a particular solution of equation

$$(t^2-1)y'' - 2ty' + 2y = 1 \ .$$

Hint: use variation of parameters.

Solution

Step 1: From question 2a, $y_1 = t$ and $y_2 = t^2 + 1$ are linearly independent solutions to the homogeneous case. We further know $W(y_1, y_2) = t^2 - 1$.

Step 2: Use the hint of variation of parameters. Rewrite the equation as

$$y'' - \frac{2t}{t^2-1}y' + \frac{2}{t^2-1}y = \frac{1}{t^2-1} =: g(t) \ .$$

Set $y_p = v_1 t + v_2(t^2+1)$. The functions v_1 and v_2 satisfy

$$v_1 = -\int \frac{(t^2+1)g(t)}{W(y_1,y_2)}dt = -\int \frac{t^2+1}{(t^2-1)^2}dt = \int \Big(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\Big)dt$$
$$= \frac{1}{2(t-1)} + \frac{1}{2(t+1)} + C_1 = \frac{t}{t^2-1} + C_1;$$

and

$$v_2 = \int \frac{tg(t)}{W(y_1,y_2)}dt = \int \frac{t}{(t^2-1)^2}dt = -\frac{1}{2(t^2-1)} + C_2.$$

Then

$$y_p = v_1y_1 + v_2y_2 = \frac{t}{t^2-1}(t) - \frac{1}{2(t^2-1)}(t^2+1) = \frac{1}{2}.$$

So $\boxed{y_p = \frac{1}{2}}$

3 [7 points]. Find a particular solution of equation

$$t^2y'' - 2ty' + 2y = t^3e^t.$$

[Bonus: 3 points] Explain whether the method of undetermined coefficients to find a particular solution of equation $t^2y'' - 2ty' + 2y = t^3e^t$ applies.

SOLUTION

Strategy: use variation of parameters.

Step 1: Find two linearly independent solutions. By making the substitution $x = \ln(t)$, we extract the characteristic equation

$$Q(r) = r^2 - (2+1)r + 2 = (r-1)(r-2)$$

Thus $y_1 = e^{2x} = t^2$ and $y_2 = e^t = t$ are linearly independent solutions to the homogeneous case.

Step 2: Set $y_p = v_1 y_1 + v_2 y_2$. Compute $W(y_1, y_2) = -t^2$. Since $g(t) = te^t$, we have

$$v_1 = -\int \frac{tg(t)}{W(y_1,y_2)}dt = -\int \frac{t \cdot te^t}{-t^2}dt = e^t + C_1$$

and

$$v_2 = \int \frac{t^2 g(t)}{W(y_1,y_2)}dt = \int \frac{t^2 \cdot te^t}{-t^2}dt = -te^t + e^t + C_2.$$

So $y_p = e^t \cdot t + (-te^t + e^t) \cdot t^2 = \boxed{y_p = te^t}$

BONUS
No: For *Euler ODE* the method of undetermined coefficients requires the right hand side to be $P(\ln t)t^m$, while it applies with $P(t)e^{mt}$ (where in both cases $P(x)$ is a polynomial in x) only when the coefficients of the ODE are constants.

4 [7 points]. Find a particular solution of equation

$$y''' - 2y'' + 4y' - 8y = e^{3x}$$

SOLUTION

Strategy: This is a constant coefficient ODE, so use the fast method shown in class. The methods of undetermined coefficients or variation of parameters will work too, but they will involve more computation.

Let $\mathcal{L} = y''' - 2y'' + 4y' - 8$. Guess of the form $y_p = A\frac{x^m}{m!}e^{3x}$, for some m. The characteristic equation of $\mathcal{L}$ is $Q(r) = r^3 - 2r^2 + 4r - 8$. Notice that $Q(3) = 27 - 18 + 12 - 8 = 13 \neq 0$.

Thus take $m = 0$. Then $\mathcal{L}\left(A\frac{x^m}{m!}e^{3x}\right) = Ae^{3x}Q(3)$. Since we want $\mathcal{L}\left(A\frac{x^m}{m!}e^{3x}\right) = e^{3x}$, we find $A = \frac{1}{13}$.

So $\boxed{y_p = \frac{1}{13}e^{3x}}$

5 [7 points]. Solve the system of ordinary differential equations

$$\begin{cases} x'_t = 5x - 3y, \\ y'_t = 6x - 4y. \end{cases}$$

SOLUTION

Step 1: Rewrite the system in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: A\bar{x}.$$

Step 2: Find the eigenvalues of A.

$$\det(A-\lambda I) = \det \begin{pmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{pmatrix} = -(5-\lambda)(4+\lambda)-(-18) = (\lambda+1)(\lambda-2)$$

So $\lambda = -1, 2$ are eigenvalues.

Step 3: Find the corresponding eigenvectors.

(1) $\lambda = -1$:

$$0 = (A+I)\bar{v} = \begin{pmatrix} 6 & -3 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 0 = 6v_1 - 3v_2.$$

So we may take $v_1 = 1$, and find $v_2 = 2$. Thus $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector.

(2) $\lambda = 2$:

$$0 = (A-2I)\bar{v} = \begin{pmatrix} 3 & -3 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 0 = v_1 - v_2.$$

So we may take $v_1 = 1$, and find $v_2 = 1$. Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

So $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

REMARK. To find the general solution of $x' = Ax$ with $n \times n$ diagonalizable matrix A it suffices to find all eigenvalues λ_j and and the corresponding eigenvectors v_j; then the answer is $\sum_j C_j e^{t\lambda_j} v_j$ where C_j $(j = 1, \dots, n)$ are constants. (It is not necessary to find T^{-1} for the matrix T whose columns are vectors v_j).